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Cosmological Mesonic Viscous Fluid Model

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A class of exact nonstatic solutions is obtained for Einstein field equations in a closed elliptic Robertson–Walker spacetime filled with viscous perfect fluid in the presence of attractive scalar fields. The solutions characterize strong interaction of elementary particles. It is also shown that the massive graviton possesses zero spin.

1. INTRODUCTION

It has been a subject of interest of cosmologists to study the nature of scalar fields with or without a mass parameter interacting with a perfect fluid distribution in order to draw an analogy of the physics of the cosmos with experimental results. It has also been established in quantum physics that a massive scalar field is associated with zero-spin chargeless particles like π- and K-mesons and the study of such fields in general relativity has therefore been extensively worked out to obtain a picture of space-time and the gravitational field associated with neutral elementary particles of zero spin.

Hawking and Ellis (1973) showed that the flat Robertson–Walker (RW) model with a massless scalar field can be reduced to a steady-state model as time $t \to \infty$. Heller and Suszycki (1974) investigated the Friedman equation with a bulk viscosity term for dust-filled models. Roy and Tiwari (1983) obtained a number of exact solutions of Einstein's equations for a viscous fluid with constant bulk viscosity. Mohanty and Pradhan (1990) investigated the problem of the interactions of a gravitational field with bulk viscous fluid in Robertson–Walker spacetime. Mohanty and Pattanaik (1991) also studied the anisotropic cosmological model with constant bulk viscous coefficient.

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In the present paper we extend our earlier work (Mohanty and Pradhan, 1990) to the case of a viscous fluid in the presence of an attractive massive scalar field. The field equations are derived and solved completely in Section 2 for a class of exact and explicit solutions. In Section 3 the energy conditions are verified. In Sections 4–9 we analyze the consequences of the results through different physical quantities involved in the solutions. The most important physical consequences are discussed in the concluding remarks in Section 10.

2. EINSTEIN'S FIELD EQUATIONS AND THEIR SOLUTIONS

Here we consider the spacetime described by an isotropic, homogeneous RW metric

$$ds^2 = dt^2 - Q^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right] \tag{1}$$

where K is the curvature index, which can take values -1, 0, $+1$, and $Q(t)$ represents the radius of the universe.

Einstein's field equations for a gravitating viscous fluid with cosmological term Λg_{ij} may be written as

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2}Rg_{ij} + \Lambda g_{ij} = -k\left(T^p_{ij} + \frac{1}{4\pi}\,T^v_{ij}\right) \tag{2}$$

where T^p_{ij} is the energy-momentum tensor due to the viscous fluid written in the form

$$T^p_{ij} = (\varepsilon + \bar{p})U_iU_j - \bar{p}g_{ij} \tag{3}$$

$$\bar{p} = p - \eta U^i_{;i} \tag{4}$$

$$U^iU_i = 1 \tag{5}$$

T^v_{ij} is the stress-energy tensor corresponding to an attractive massive scalar field, given by

$$T^v_{ij} = \frac{1}{4\pi}\{V_{,i}V_{,j} - \tfrac{1}{2}g_{ij}(V_{,k}V^{,k} - M^2V^2)\} \tag{6}$$

Hereafter, a comma and a semicolon denote ordinary and covariant differentiation, respectively.

The scalar field V satisfies the Klein–Gordon equation

$$g^{ij}V_{,ij}+M^2V=0 \tag{7}$$

ε is the energy density, p is the pressure, and η is the bulk viscous coefficient of the distribution.

For the metric (1) the set of field equations (2) in a comoving coordinate system, i.e., $U_i=\delta_0^i$ reduces to the following explicit forms:

$$\begin{aligned}G_{11}&\equiv K+Q_4^2+2QQ_{44}-\Lambda Q^2\\&=-k\left\{\bar{p}Q^2+\frac{1}{8\pi}\left[V_1^2(1-Kr^2)+Q^2(V_4^2-M^2V^2)\right]\right\}\end{aligned} \tag{8}$$

$$\begin{aligned}G_{22}&\equiv K+Q_4^2+2QQ_{44}-Q^2\\&=-k\left\{\bar{p}+\frac{1}{8\pi}[-(1-Kr^2)V_1^2+Q^2(V_4^2-M^2V^2)]\right\}\end{aligned} \tag{9}$$

$$G_{33}=G_{22}$$

and

$$\begin{aligned}G_{44}&\equiv Q^2-3Q_4^2-3K\\&=-k\left\{Q^2+\frac{1}{8\pi}\left[(1-Kr^2)V_1^2+Q^2(V_4^2+M^2V^2)\right]\right\}\end{aligned} \tag{10}$$

where

$$V_{,2}=V_{,3}=0 \tag{11}$$

Hereafter, the indexes 1 and 4 after a field variable denote partial differentiation with respect to r and t, respectively. The velocity of light is chosen to be unity throughout the discussion.

Here one can easily get from (8) and (9) that

$$V_1=0 \tag{12}$$

Now the Klein–Gordon equation (7) for the metric (1) becomes

$$V_{44}+3\frac{Q_4V_4}{Q}+M^2V=0 \tag{13}$$

Further, equations (8), (11), and (12) yield

$$Q_{44} - \tfrac{1}{3}\Lambda Q = -\frac{kQ}{6}\left[(\varepsilon + 3\bar{p}) + \frac{1}{4\pi}(2V_4^2 - M^2V^2)\right] \tag{14}$$

In order to avoid the mathematical complexity due to the highly nonlinear nature of the field equations, we assume

$$2V_4^2 - M^2V^2 = 0 \tag{15}$$

On integration, equation (15) yields either

$$V = A\, e^{(M/\sqrt{2})t} \tag{16a}$$

or

$$V = B\, e^{-(M/\sqrt{2})t} \tag{16b}$$

where A and B are arbitrary constants of integration.

Now with the aid of equation (12) one can obtain the expression for ε and $\bar{p}$ from (11) and (8) as

$$\varepsilon = \frac{1}{k}\left(-\Lambda + 3\frac{Q_4^2}{Q^2} + 3\frac{K}{Q^2}\right) - \frac{1}{8\pi}(V_4^2 + M^2V^2) \tag{17}$$

and

$$\bar{p} = \frac{1}{k}\left[\Lambda - \frac{K}{Q^2} - \left(\frac{Q_4}{Q}\right)^2 - \frac{2Q_{44}}{Q} - \frac{2Q_{44}}{Q}\right] - \frac{1}{8\pi}(V_4^2 - M^2V^2) \tag{18}$$

Now using (16) in (13), one can get either

$$Q = C\, e^{-(M/\sqrt{2})t} \tag{19a}$$

or

$$Q = D\, e^{(M/\sqrt{2})t} \tag{19b}$$

where C and D are arbitrary constants of integration.

Using the values of V and Q from equations (16a) and (19a) in equations (17) and (18), we obtain the following explicit expressions for ε and $\bar{p}$:

$$\varepsilon = \frac{1}{k}(-\Lambda + \tfrac{3}{2}M^2) + 3\left(\frac{K}{C^2k} - \frac{A^2M^2}{16\pi}\right)e^{\sqrt{2}Mt} \tag{20}$$

and

$$\bar{p}=\frac{1}{k}(\Lambda-\tfrac{3}{2}M^2)+\left(\frac{A^2M^2}{16\pi}-\frac{K}{C^2k}\right)e^{\sqrt{2}Mt} \tag{21}$$

If the distribution is restricted with the baryotropic equation of state, i.e.,

$$p=(\gamma-1)\varepsilon, \qquad 0\leq\gamma\leq 2 \tag{22}$$

one can obtain the physical quantity η as

$$\eta=\frac{\sqrt{2}\,\gamma}{3Mk}(\Lambda-\tfrac{3}{2}M^2)+\frac{\sqrt{2}}{3M}(2-3\gamma)\left(\frac{K}{C^2k}-\frac{A^2M^2}{16\pi}\right)e^{\sqrt{2}Mt} \tag{23}$$

The physical quantities corresponding to the alternative case given by (16b) and (19b) can be obtained as

$$\varepsilon=\frac{1}{k}(-\Lambda+\tfrac{3}{2}M^2)+3\left(\frac{K}{kD^2}-\frac{B^2M^2}{16\pi}\right)e^{-\sqrt{2}Mt} \tag{24}$$

and

$$\eta=\frac{\sqrt{2}\,\gamma}{3Mk}(-\Lambda+\tfrac{3}{2}M^2)+\frac{\sqrt{2}}{3M}(3\gamma-2)\left(\frac{K}{kD^2}-\frac{B^2M^2}{16\pi}\right)e^{-\sqrt{2}Mt} \tag{25}$$

The metrics corresponding to the solutions (19a) and (19b) can be written as

$$ds^2=dt^2-C^2\,e^{-\sqrt{2}Mt}\left(\frac{dr^2}{1-Kr^2}+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2\right) \tag{26}$$

and

$$ds^2=dt^2-D^2\,e^{\sqrt{2}Mt}\left(\frac{dr^2}{1-Kr^2}+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2\right) \tag{27}$$

3. ENERGY CONDITIONS FOR VISCOUS FLUID

The strong energy condition for both solutions leads to the viscous analogue of gravitational mass density (McCrea, 1951), i.e.,

$$\sigma_1=\rho+3\bar{p}=\frac{1}{k}(2\Lambda-3M^2)>0 \tag{28}$$

It is clear from the above expression that $\sigma_1>0$ only when $\Lambda>\frac{3}{2}M^2$. This indicates that the model is physically acceptable only when the cosmological constant $\Lambda>0$.

The weak energy condition, i.e., $\sigma_2 = (\varepsilon + \bar{p}) > 0$, corresponding to the solutions (26) and (27) leads to the following relations between the arbitrary constants involved in the solutions:

$$16\pi K > A^2 M^2 C^2 k \tag{29a}$$

$$16\pi K > B^2 M^2 D^2 k \tag{29b}$$

As the quantities involved in the right side of the above expressions are positive, the only physically realistic models acceptable are closed elliptic models with $K = 1$. Moreover, it appears that the other open models with $K = 0, -1$ may be possible in inflationary cosmological models of the grand unified phase transition.

However, the model violates the strong energy condition for $\Lambda \leq 0$ and this situation may also correspond to a grand unified phase transition.

4. EXPANSION SCALAR AND SHEAR

4.1. Expansion Scalar θ

The values of the expansion scalar $\theta = U^i_{;i} = 3Q_4/Q$ for solutions (26) and (27) are

$$\theta = \pm \frac{3}{\sqrt{2}} M \tag{30}$$

respectively. These results clearly indicate that the models (26) and (27) correspond to contracting and expanding models, respectively, subject to the reality assumption of positivity of mass parameter associated with the scalar field.

4.2. Shear σ

The scalar shear $\sigma = \frac{1}{2}\sigma_{ij}\sigma^{ij}$ for the models (26) and (27) is given by

$$\sigma = 16\left(\frac{Q_4}{Q}\right)^2 = 8M^2 \tag{31}$$

Here, σ being positive indicates that the shape of the universe is preserved during the evolution in the case of both models. However, both models do not admit rotation (i.e., $W_{ij} = 0$).

5. BULK VISCOUS COEFFICIENT η

In the case of the contracting model represented by equation (26), the bulk viscous coefficient increases with time and we get

$$\text{as} \quad t \to \infty, \qquad \eta \to \pm\infty \quad \text{according as} \quad \gamma \lessgtr \tfrac{2}{3}$$

whereas in the case of the expanding model corresponding to the solution (27), we have

$$\text{as} \quad t \to \infty, \qquad \eta \to \text{const} < 0$$

However, in both cases the solutions lead to unphysical situations at infinite future. In view of energy conditions, η is negative for $\gamma = 2/3$ in the expanding model, which leads to an unphysical situation. But the corresponding situation in the contracting model is physically acceptable with constant bulk viscous coefficient, i.e.,

$$\eta = \eta_0 = \frac{\sqrt{2}}{9Mk} (2\Lambda - 3M^2)$$

As $\eta > 0$, the models are physically acceptable at the initial epoch if

$$(\sigma_1)_{t=0} > \frac{3\gamma - 2}{\gamma} (\sigma_2)_{t=0}$$

6. ENERGY DENSITY ε

Both models are physically realistic at the initial epoch provided that $\varepsilon > 0$, which yields that the weak energy condition dominates over the strong energy condition, i.e.,

$$3(\sigma_2)_{t=0} > (\sigma_1)_{t=0}$$

In a mixture of mesonic viscous fluid the big bang singularity does not occur at the initial epoch. This situation is similar to that of the case studied earlier by Mohanty and Pradhan (1990) in the case of a viscous fluid alone.

In the case of the contracting model represented by equation (26) there may be a big crunch at infinite future, since $\varepsilon \to \infty$ as $t \to \infty$. In the case of the expanding model we have an unphysical situation because

$$\text{as} \quad t \to \infty, \qquad \varepsilon \to a \ (=\text{const}) < 0$$

Thus, both models lead to unphysical situations at infinite future.

7. MASSIVE SCALAR FIELD V

Here it is sufficient to consider the solution given by (16b) and (19b) since the analysis of the metric is identical for both solutions, except that the solution given by (16b) and (19b) represents an expanding model, while (16a) and (19a) represent a contracting model. It can be clearly observed for the expanding spacetime (27) that the scalar field V decreases exponentially with time. The energy density associated with the scalar field is given as (Anderson, 1967, p. 289) as

$$\rho=\tfrac{1}{2}(V_4^2+M^2V^2)$$

From equation (16b), we have

$$\rho=(3/4)B^2M^2\,e^{-\sqrt{2}Mt}$$

so that, in expanding spacetime (27), the energy density of the scalar field also decreases with time, but at a faster rate than the scalar V. For physically acceptable mesonic field we have $\rho>0$.

8. HUBBLE PARAMETER H AND DECELERATION PARAMETER q

Corresponding to the metric (27), the value of the Hubble parameter H is

$$H=\frac{Q_4}{Q}=\frac{M}{\sqrt{2}}$$

and the value of the deceleration parameter q is

$$q=-\frac{Q_{44}Q}{Q_4^2}=-1$$

Thus, the values of H and q might seem to suggest that the spacetime represents a steady-state model, but this is not so, because in a steady-state model the curvature index K is necessarily zero (Weinberg, 1972, p. 459). Moreover, we found earlier that models represented by either (26) or (27) are physically realistic only when $K=1$.

9. THE PARAMETER γ

For many realistic physical situations it is required that $\varepsilon\geq p\geq 0$, which yields restrictions on the parameter γ, i.e., $1\leq\gamma\leq 2$. These restrictions include the extreme cases corresponding to the viscous-dust ($\varepsilon>p=0$) and stiff-perfect-fluid ($\varepsilon=p>0$) models. It is interesting to note that the spacetime does not have any singularity in any finite epoch and one of the alternative cases corresponding to $\varepsilon = p = 0$ removes the viscous fluid from the cosmological models obtained earlier.

10. CONCLUSION

It is well known that the massive scalar field, considered in the present investigation with mass parameter M, is related to the mass m of a zero-spin particle

$$M = \frac{m}{\hbar}$$

When the mass of the scalar field corresponds to a neutral meson, the Hubble constant H for the solution (27) becomes

$$H = \frac{M}{\sqrt{2}} = \frac{m_\pi}{\sqrt{2}\,\hbar} = \frac{m_\pi}{\sqrt{2}\,\hbar} \simeq 2 \times 10^{22}\ \mathrm{sec}$$

where m_π is the mass of the neutral π-meson.

The characteristic time corresponding to the value of Hubble parameter is given by

$$T = H^{-1} \simeq 5 \times 10^{-23}\ \mathrm{sec}$$

This value of the characteristic time corresponds to the strong interaction associated with elementary particles (Isham *et al.*, 1971). However, in the cosmological case

$$H \simeq \frac{mg}{\hbar}$$

where the velocity of light is unity. We conclude that the mass parameter involved in the Klein–Gordon equation corresponding to the mixture of mesonic viscous fluids is the mass of the graviton. Thus, this indicates that besides the viscous perfect fluid, the spin-zero graviton is also responsible for cosmological effects. Gursey (1963) showed that this spin-zero graviton yields cosmological effects, while the spin-two graviton yields gravitational attraction.

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